The Open University

M386/Specimen

Third Level Course Examination

Metric and Topological Spaces and Geometric Topology

Time allowed: 3 hours

This paper is in two Parts. Part I (Questions 1–8) is on Metric and Topological Spaces, and Part II (Questions 9–16) is on Geometric Topology.

Your examination grade will be the sum of your best **NINE** question scores, where *not more than six* of these come from a single Part. If you exceed six questions from one Part, all questions will be marked but credit will be given only for your best **NINE** questions within the restriction stated above. You may cross out any work that you do not wish the examiner to mark.

Use a **separate** answer book for **each** part.

All questions carry equal marks. The allocation of marks within a question is indicated by a number in brackets, [ ], beside the question.

At the end of the examination

Check that you have written your name, personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.**

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2.2

Answer as many questions as you wish. Full marks may be obtained by **complete** *answers to* **NINE** *questions, provided that no more than* **SIX** *questions have been selected from any one part. All questions carry equal marks.*

PART I METRIC AND TOPOLOGICAL SPACES

Question 1

(i) Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$ and let $f : A \to B$ be given by

$$f(a) = f(b) = 2, \quad f(c) = 3.$$

Write down:

(a) $f(f^{-1}(\{1, 2\}))$;

(b) $f(f^{-1}(\{2, 3\}))$. [4]

(ii) Let $f : A \to B$ be a function, where A and B are non-empty sets.

(a) Prove that if $D \subset B$ then

$$f(f^{-1}(D)) \subset D.$$ [3]

(b) Prove that if f is an onto function and $D \subset B$ then

$$D = f(f^{-1}(D)).$$ [4]

Question 2

Let $d : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be defined by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ |x - 1| + |y - 1|, & \text{if } x \neq y. \end{cases}$$

(i) Show that d is a metric on $\mathbf{R}$. [5]

(ii) Determine the following open balls with respect to the metric d:

(a) $B_1(0)$;

(b) $B_1(1)$. [3]

(iii) Show that the function $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = 1 - x, \qquad x \in \mathbf{R}$$

is not (d, d)-continuous at 1. [3]

Question 3

Let E be the set of even positive integers and let $\mathcal{T}$ be the collection of subsets of the set $\mathbf{N}$ of positive integers defined as follows: if $V \subset \mathbf{N}$ then $V \in \mathcal{T}$ if $1 \notin V$ or $E \subset V$.

(i) Show that $\mathcal{T}$ is a topology on $\mathbf{N}$. [7]

(ii) Determine the topology induced by $\mathcal{T}$ on:

(a) the set of odd positive integers; [2]

(b) $\{1, 2\}$. [2]

Question 4

(i) Let T_1 and T_2 be topological spaces and let $\mathcal{B}$ be a basis for the topology of T_2. Prove that a function $f : T_1 \longrightarrow T_2$ is continuous if $f^{-1}(B)$ is open in T_1 for each set B in $\mathcal{B}$. [4]

(ii) Let $T_1 = \{A, \mathcal{J}_1\}$ where $A = \{a, b, c\}$ and

$$\mathcal{J}_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, A\}.$$

[You need not verify that $\mathcal{J}_1$ is a topology on A.]

Prove that if T_2 denotes $\mathbf{R}$ with the usual topology, then a function $g : T_1 \to T_2$ is continuous if and only if $g(b) = g(c)$. [7]

Question 5

For each positive integer m, let

$$B_m = \{1, 3, \ldots, 2m - 1\}.$$

(i) Prove that the collection $\mathcal{B}$ of subsets of the set $\mathbf{N}$ of positive integers which consists of $\mathbf{N}$ together with the sets B_m, $m \in \mathbf{N}$, is a synthetic basis for $\mathbf{N}$. [4]

(ii) Prove that the set E of all even positive integers is closed in the topological space $\{\mathbf{N}, \mathcal{T}\}$, where $\mathcal{T}$ is the topology on $\mathbf{N}$ arising from $\mathcal{B}$ in part (i). [3]

(iii) Determine the closure in $\{\mathbf{N}, \mathcal{T}\}$ of the set $\{2, 6, 7, 8\}$. [4]

Question 6

(i) Prove that each compact subspace of a Hausdorff space T is closed in T. [5]

(ii) For each of the following subspaces of $\mathbf{R}^2$ with the usual topology, state whether or not the subspace is compact. In each case give a brief justification of your answer. [6]

(a) $\{(x_1, 0); x_1 \leqslant 0\}$

(b) $\{(x_1, x_2) : x_1^2 + x_2^2 \leqslant 3\}$

(c) $([0, 1] \times [0, 1]) \cap (Q \times Q)$, where Q is the set of rationals.

Question 7

(i) Assuming that the subspace [0,1] of $\mathbf{R}$ with the usual topology is connected, prove that every path-connected space is connected. [5]

(ii) For each positive integer n, let L_n be the line segment in $\mathbf{R}^2$ joining $\left(\frac{1}{n}, 0\right)$ to $\left(\frac{1}{n}, 1\right)$, let

$$H = \{(x_1, x_2) \in \mathbf{R}^2 : 0 < x_1 \leqslant 1, x_2 = 0\},$$

and let

$$K = \{(0, 1)\} \cup H \cup \bigcup_{n=1}^{\infty} L_n.$$

Prove that the subspace K of $\mathbf{R}^2$ with the usual topology is connected. [6]

Question 8

(i) Prove that a complete subspace of a metric space M is closed in M. [4]

(ii) Let d be the metric on the set $\mathbf{N}$ of positive integers defined by

$$d(i,j) = \left|\frac{1}{i} - \frac{1}{j}\right|, \qquad i,j \in \mathbf{N}.$$

(a) Prove that if $x_n = 2^n$, $n \in \mathbf{N}$, then (x_n) is a Cauchy sequence in the metric space $\{\mathbf{N}, d\}$. [3]

(b) Show that if $k \in \mathbf{N}$, then

$$d(k, x_n) > \frac{1}{2k},$$

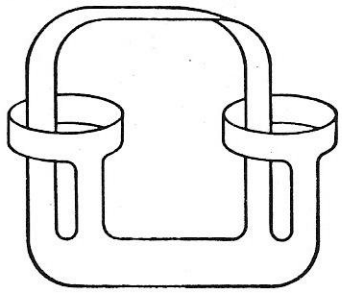
for each integer n such that $n \geqslant 2k$. [2]

(c) Deduce that the metric space $\{\mathbf{N}, d\}$ is not complete. [2]

Question 9

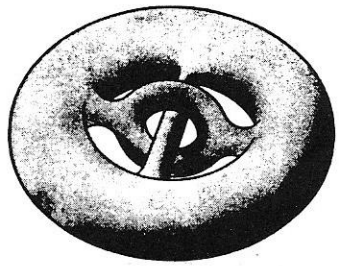
Redraw each of the following surfaces in a simpler form, labelling clearly any cutting and glueing operations. Hence write down χ and β for each surface, and whether or not each is orientable.

(i) (thin paper)



[6]

(ii) (tubular)



[5]

Question 10

(i) Write down the conditions that a closed surface, subdivided into F heptagons (7-sided polygons), has k faces meeting at each vertex. (The number of edges and vertices are E and V respectively.) [3]

(ii) Show that no surface of Euler characteristic $\chi = -2$ can be subdivided as in (i) if $k = 8$. [4]

(iii) Show that no surface of Euler characteristic $\chi = -2$ can be subdivided as in (i) unless $k = 3$, $F = 12$. [4]

Question 11

Reduce each of the following systems of edge equations to canonical form. Justify your working. Hence calculate χ and β for the surfaces they represent.

(i) $ab = 1$; $bc = 1$; $cd = 1$. [3]

(ii) $abacbdcd = 1$. [4]

(iii) $aba^{-1}b^{-1}cc = 1$. [4]

Question 12

Determine all the distinct surfaces of Euler characteristic

(i) 2,

(ii) 1,

(iii) 0,

(iv) -1,

using connected sum notation.

For each surface, write down the number of its boundary curves and state whether or not it is orientable. [11]

Question 13

Categorize each of the following conics (which you may assume are non-degenerate) as one of the following:

(a) homeomorphic to the z-sphere;
(b) a double cover of the z-sphere branched over two points;
(c) a double cover of the z-sphere branched over two points and having a pinchpoint.

In each case locate the branch points and pinch points, where they exist.

(i) $z^2 + 2zw + 6 = 0$ [3]

(ii) $z^2 + 2zw + 4w^2 + 6 = 0$ [4]

(iii) $z^2 + 2zw + w^2 + 6z = 0$ [4]

Question 14

(i) Write down the Euler characteristic of an orientable surface of genus g with b boundary components. [2]

(ii) Let S be an orientable surface of genus g with 2 boundary curves, and let $\tilde{S}$ be an orientable surface of genus $\tilde{g} = g + k$ with $\tilde{b}$ boundary curves. Suppose $\tilde{S}$ is an unbranched n-fold cover of $S (n > 1)$. Using the inequalities relating $\tilde{b}$ to b and nb, show that

$$(n-1)(g-1) \leqslant k \leqslant (n-1)g.$$ [6]

(iii) Deduce that if $n = 3$ and $g > 2$, then $\tilde{g}$ cannot be $2g$. [3]

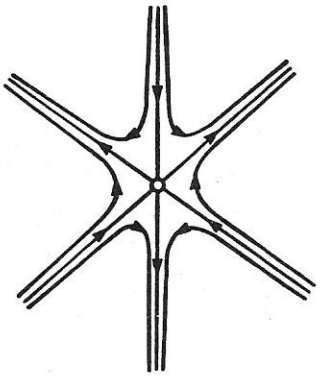
Question 15

(i) Show that the zeros of $F(z) = z^4 + 2z + 5$ all lie in the annulus $1 \leqslant |z| \leqslant 2$. [5]

(ii) Show that F has no real or imaginary zeros, and that its zeros are distributed one in each quadrant. [6]

Question 16

(i) Find the index of the rest point in the flow field sketched below. [7]



(ii) Explain *briefly* why only one closed orientable surface can have a flow field with this as its only rest point. [4]

[END OF QUESTION PAPER]

THE OPEN UNIVERSITY

Mathematics: A Third Level Course

M386 Metric and Topological Spaces and Geometric Topology

M386 Solutions to Specimen Examination Paper

Part I Metric and Topological Spaces

Question 1

(i) (a) $f^{-1}(\{1,2\}) = \{a,b\}$ so that $f(f^{-1}\{1,2\})) = \{2\}$.

(b) $f^{-1}(\{2,3\}) = \{a,b,c\}$ so that $f(f^{-1}(\{2,3\})) = \{2,3\}$.

(ii) (a) If $y \in f(f^{-1}(D))$, then $y = f(x)$ where $x \in f^{-1}(D)$. But since $x \in f^{-1}(D)$ it follows that $y = f(x) \in D$. Hence

$$f(f^{-1}(D)) \subset D.$$

(b) If $y \in D$ then since f is onto there exists $x \in A$ such that $f(x) = y$. Since $f(x) \in D$ it follows that $x \in f^{-1}(D)$.Thus $y = f(x)$ where $x \in f^{-1}(D)$ and so $y \in f(f^{-1}(D))$. Thus $D \subset f(f^{-1}(D))$ so that from (a) we have

$$D = f(f^{-1}(D)).$$

Question 2

(i) We must check that (M1), (M2) and (M3) are satisfied.

M1 If $x = y$, then $d(x,y) = 0$.
If $x \neq y$, then either $x \neq 1$ or $y \neq 1$ so $d(x,y) > 0$.

M2 is obviously satisfied.

M3 We need only verify that the triangle inequality holds for x, y, z all different.

$$\begin{aligned} d(x,y) + d(y,z) &= |x-1| + 2|y-1| + |z-1| \\ &\geqslant |x-1| + |z-1| = d(x,z), \end{aligned}$$

since $|y-1| \geqslant 0$.

Since d satisfies (M1), (M2) and (M3), it is a metric on $\mathbf{R}$.

(ii) (a) If $y \neq 0$, then $d(0,y) = 1 + |y-1|$. There is no y such that $1 + |y-1| < 1$ and so $B_1(0) = \{0\}$.

(b) Since $d(1,y) = |y-1|$, we see that $B_1(1) = (0,2)$.

(iii) Suppose f is (d,d)-continuous at 1. Then there exists $\delta > 0$ such that $f(x) \in B_1(0)$ if $x \in B_\delta(1)$. Thus there exists $\delta > 0$ such that $f(x) = 0$ if $x \in (1-\delta, 1+\delta)$. Since $f(1 - \frac{1}{2}\delta) = \frac{1}{2}\delta \neq 0$, we have a contradiction. Hence f is not (d,d)-continuous at 1.

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Question 3

(i) We must show that (T1), (T2) and (T3) are satisfied.

T1 Since $1 \notin \emptyset$, $\emptyset \in \mathcal{T}$.
Since $E \subset \mathbf{N}$, $\mathbf{N} \in \mathcal{T}$.

T2 Let $V_1, V_2 \in \mathcal{T}$. If $1 \notin V_1 \cap V_2$ then $V_1 \cap V_2 \in \mathcal{T}$. If $1 \in V_1 \cap V_2$ then $1 \in V_1$ and $1 \in V_2$. Since $V_1 \in \mathcal{T}$ and $1 \in V_1$ it follows that $E \subset V_1$. Similarly $E \subset V_2$. Thus $E \subset V_1 \cap V_2$ and so $V_1 \cap V_2 \in \mathcal{T}$.

T3 Let $(V_i)_{i \in I}$ be a family of elements of $\mathcal{T}$ and let $V = \bigcup_{i \in I} V_i$. If $1 \notin V$ then $V \in \mathcal{T}$. If $1 \in V$ then there exists $j \in I$ such that $1 \in V_j$. Since $V_j \in \mathcal{T}$ and $1 \in V_j$ it follows that $E \subset V_j$. Hence $E \subset V_j \subset V$ and so $V \in \mathcal{T}$.

(ii) (a) Put $H = \{n \in \mathbf{N} : n \text{ is odd}\}$. If $A \subset H$, then $A \cup E \in \mathcal{T}$. Since $(A \cup E) \cap H = A$, we see that A belongs to the topology induced on H by $\mathcal{T}$. Thus the topology induced on H by $\mathcal{T}$ is the discrete topology.

(b) Put $K = \{1, 2\}$ and let $\mathcal{T}_1$ be the topology induced on K by $\mathcal{T}$. Suppose $W \in \mathcal{T}_1$ and $1 \in W$. Then there exists $V \in \mathcal{T}$ such that $V \cap K = W$. Since $1 \in V$, it follows that $E \subset V$ and so $W = V \cap K = K$. Since $\{2\} = \{2\} \cap K$ and $\{2\} \in \mathcal{T}$ we see that

$$\mathcal{T}_1 = \{\emptyset, \{2\}, K\}.$$

Question 4

(i) Let V be open in T_2. Then there is a collection $(B_i)_{i \in I}$ of members of $\mathcal{B}$ much that $V = \bigcup_{i \in I} B_i$. Thus

$$f^{-1}(V) = f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i).$$

Since $f^{-1}(B_i)$ is open in T_1 for each i, it follows that $f^{-1}(V)$ is open in T_1.

(ii) Suppose first that $g(b) \neq g(i)$ and put $W = \mathbf{R} - \{g(b)\}$. Then W is open in T_2 and since $g(c) \in W$ and $g(b) \notin W$, it follows that $c \in g^{-1}(W)$ and $b \notin g^{-1}(W)$. But the open sets of T_1 which contain c are $\{b, c\}$ and A. Thus $g^{-1}(W)$ is not open in T_1. Hence $g : T_1 \longrightarrow T_2$ is not continuous.
Now suppose that $g(b) = g(c) = t$, say. If V is open in T_2, then

$$g^{-1}(V) = \begin{cases} \emptyset \text{ or } \{a\} & \text{if } t \notin V \\ \{b, c\} \text{ or } A & \text{if } t \in V \end{cases}$$

Hence $g^{-1}(V)$ is open in T_1. Therefore $g : T_1 \longrightarrow T_2$ is continuous.

Question 5

(i) Since $\mathbf{N} \in \mathcal{B}$, (B1) is satisfied.
If $m, n \in \mathbf{N}$ and $m < n$ then $B_m \cap B_n = B_m$. Thus (B2) is satisfied.
Since (B1) and (B2) are satisfied, $\mathcal{B}$ is a synthetic basis for $\mathbf{N}$.

(ii) $\bigcup_{m=1}^{\infty} B_m$ is open in $\{\mathbf{N}, \mathcal{T}\}$. But $\bigcup_{m=1}^{\infty} B_m$ is the set of odd positive integers,
Thus

$$E = \mathbf{N} - \bigcup_{m=1}^{\infty} B_m$$

is closed in $\{\mathbf{N}, \mathcal{T}\}$.

(iii) Put $H = \{2, 6, 7, 8\}$. If $k \in \mathbf{N}$ and k is even, then $\mathbf{N}$ is only open set which contains k and so k is a limit point of H. If $m > 4$ then every open set which contains $2m-1$ contains B_m; since $B_m \cap H = \{7\} \neq \emptyset$, $2m-1$ is a limit point of H. Thus

$$Cl(H) = (H) \cup \{n \in \mathbf{N} : n = 4 \text{ or } n \geqslant 9\}.$$

Question 6

(i) **S** Proposition 5.4.2, page 82.

(ii) (a) Not compact, as the subspace is not bounded.

(b) Compact, as the subspace is closed and bounded.

(c) Not compact, as the subspace is not closed.

Question 7

(i) **S** Proposition 6.4.1, page 102.

(ii) The subspace $A = H \cup \bigcup_{n=1}^{\infty} L_n$ is path-connected and hence connected. Let $\varepsilon > 0$ be given, and choose N such that $\frac{1}{N} < \varepsilon$.

Then $\left(\frac{1}{N}, 1\right) \in B_\varepsilon((0,1)) \cap L_N$. Thus $(0,1) \in Cl(A)$. Since $A \subset K \subset Cl(A)$ and A is connected, it follows from **S** Proposition 6.2.18, page 99, that K is connected.

Question 8

(i) **S** Proposition 9.2.3(a), page 123.

(ii) (a) If $m, n \in \mathbf{N}$ and $m < n$ then

$$d(x_n, x_m) = \left|\frac{1}{2^n} - \frac{1}{2^m}\right| = \frac{1}{2^m} - \frac{1}{2^n} < \frac{1}{2^m}.$$

Let $\varepsilon > 0$ be given. There exists a positive integer N such that $\frac{1}{2^N} < \varepsilon$. If $m, n \in \mathbf{N}$ and $n > m \geqslant N$ then

$$d(x_n, x_m) < \frac{1}{2^m} \leqslant \frac{1}{2^N} < \varepsilon.$$

Thus (x_n) is a Cauchy sequence in $\{\mathbf{N}, d\}$.

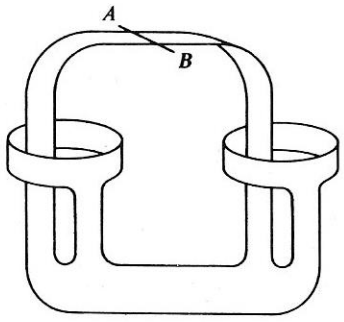
(b) If $n \geqslant 2k$ then

$$d(k, x_n) = \left|\frac{1}{k} - \frac{1}{2^n}\right| = \frac{1}{k} - \frac{1}{2^n} \geqslant \frac{1}{k} - \frac{1}{2^{2k}} > \frac{1}{k} - \frac{1}{2k} = \frac{1}{2k}.$$

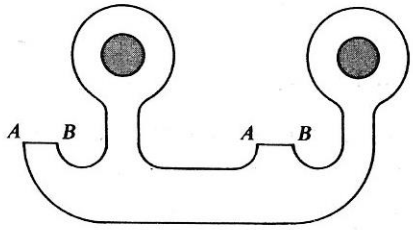
(c) Suppose that $\{\mathbf{N}, d\}$ is complete. Then the Cauchy sequence (x_n) is convergent. Suppose that (x_n) converges to $k \in \mathbf{N}$. Then there exists a positive integer N such that $d(k, x_n) < \frac{1}{2k}$ for all $n \geq N$. But there exists $n \in \mathbf{N}$ such that $n \geqslant N$ and $n \geqslant 2k$. We then have a contradiction of (b). Therefore $\{\mathbf{N}, d\}$ is not complete.

Question 9

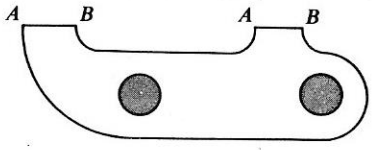
(i)



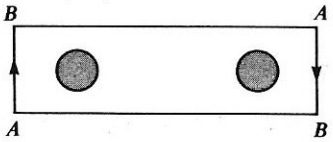
Cut (for example) at AB. Retracting the arms and flattening the cylinders into annuli gives the following figure.



Pulling the 'holes' inside gives

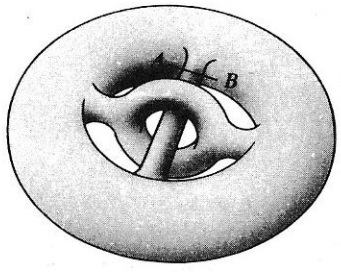


and this is now recognizable as a Möbius band with two holes in it.

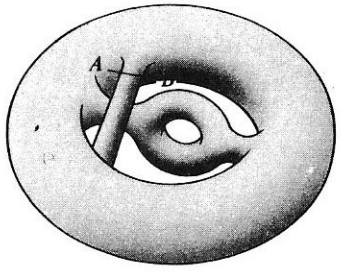


$\beta(M^2) = 1$; $\chi(M^2) = 0$; so $\beta(S) = 3$, $\chi(S) = -2$. S is non-orientable.

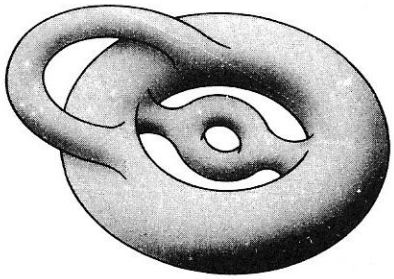
(ii)



Cutting, for example, at AB, retracting the severed ends, and rejoining them outside the link, gives the following.



Now pulling the arm containing AB over to the left gives



which is a 4-holed torus.
$\chi(S) = -6$, $\beta(S) = 0$ and S is orientable.

Question 10

(i) Each of the F faces has seven edges, but each edge occurs in exactly two faces, so

$$7F = 2E.$$

At each vertex there are k faces, and so k edges. Thus (since each edge has two vertices)

$$\begin{aligned} kV &= 2E \\ &= 7F. \end{aligned}$$

(ii) $F - E + V = \chi = -2$.
Therefore

$$\left(\frac{2}{7} - 1 + \frac{2}{k}\right) E = -2.$$

If $k = 8$, then

$$E = -2 \Big/ \left(\frac{2}{7} - 1 + \frac{1}{4}\right) = \frac{56}{13} \text{ which is not an integer.}$$

This is impossible.

(iii) $\dfrac{2}{k} = \dfrac{5}{7} - \dfrac{2}{E}$, so

$$k = \frac{14E}{5E - 14}, \text{ and } E = \frac{14k}{5k - 14}.$$

The second formula shows that $k \geq 3$, since $E > 0$. Also, since there must be at least one face, $E \geqslant 7$. Thus

$$\frac{14k}{5k - 14} \geqslant 7,$$

$$\text{and then } 14k \geqslant 35k - 98,$$

$$\text{or } 21k \leqslant 98.$$

Hence $k \leqslant 4$.
But if $k = 4$ then $E = \frac{56}{6}$ which is not an integer. If $k = 3$ then a genuine possibility exists with

$$E = 42, F = 12, V = 28.$$

Question 11

(i) $ab = 1; bc = 1; cd = 1 \longrightarrow ab = 1; bd^{-1} = 1$ (eliminating c)
$\longrightarrow ad = 1$ (eliminating b)
$\longrightarrow x = 1$ (writing x for ad)
$\longrightarrow xd^{-1}d = 1$ (introducing the pair $d^{-1}d$)
$\longrightarrow dxd^{-1} = 1$ (cycling)

As the surface is orientable, $\chi = 1$ and $\beta = 1$.

(ii) $abacbdcd = 1 \longrightarrow aab^{-1}cbdcd = 1$ (Lemma 1)
$\longrightarrow aab^{-1}d^{-1}b^{-1}ccd = 1$ (Lemma 1)
$\longrightarrow aadb^{-1}b^{-1}ccd = 1$ (Lemma 1)
$\longrightarrow aac^{-1}c^{-1}bbdd = 1$ (Lemma 1)

As the surface is non-orientable, $\chi = -2$ and $\beta = 0$.

(iii) $aba^{-1}b^{-1}cc = 1 \longrightarrow abcbac = 1$ (Lemma 1)
$\longrightarrow abbc^{-1}ac = 1$ (Lemma 1)
$\longrightarrow cabbac = 1$ (Lemma 1)
$\longrightarrow abbacc = 1$ (cycling)
$\longrightarrow bbaacc = 1$ (Lemma 1, twice)

As the surface is non-orientable, $\chi = -1$ and $\beta = 0$.

Question 12

(i) The only surface for $\chi = 2$ is the sphere S^2, with $\beta = 0$. It is orientable.

(ii) Now $\chi(pT^2 \# bD^2) = 2 - 2p - b$,
$\chi(q\mathbf{RP}^2 \# bD^2) = 2 - q - b$.

$\chi = 1$ If $2 - 2p - b = 1$, then $2p + b = 1$ and $p, b \geq 0$.
Hence $p = 0, b = 1$ and we have $S^2 \# D^2 = D^2 : \beta = 1$ and it is orientable.
If $2 - q - b = 1$ then $q + b = 1$ and $b \geqslant 0, q > 0$ give $q = 1, b = 0$.

Thus we have $\mathbf{RP}^2 : \beta = 0$ and it is non-orientable.

(iii) $\chi = 0$ If $2 - 2p - b = 0$, then $2p + b = 2$.
So $p = 1, b = 0$ or $p = 0, b = 2$.
This gives $T^2(\beta = 0,$ orientable) or $2D^2(\beta = 2,$ orientable).
If $2 - q - b = 0$, then $q = 2, b = 0$ or $q = 1, b = 1$.

This gives $2\mathbf{RP}^2(\beta = 0,$ non-orientable) or $\mathbf{RP}^2 \# D^2$ $(\beta = 1,$ non-orientable).

(iv) $\chi = -1$ If $2 - 2p - b = -1$, then $2p + b = 3$.
Therefore $p = 0, b = 3$ or $p = 1, b = 1$.
This gives $3D^2(\beta = 3,$ orientable) or $T^2 \# D^2(\beta = 1,$ orientable).

If $2 - q - b = -1$, then $q + b = 3$.

Thus $q = 1, b = 2$ or $q = 2, b = 1$ or $q = 3, b = 0$.

This gives $\mathbf{RP}^2 \# 2D^2$ $(\beta = 2,$ non-orientable), $2\mathbf{RP} \# D^2$ $(\beta = 1,$ non-orientable) and $3\mathbf{RP}^2$ $(\beta = 0,$ non-orientable).

Question 13

(i) $w = -\dfrac{(z^2+6)}{2z}$ gives a $1-1$ correspondence of the conic with the z-sphere.
The conic is a sphere with no branch point or pinch point.

(ii) $A=1$, $B=2$, $C=4$ and so by the main theorem we know that the conic is a pinched sphere.

$$w = \frac{-2z \pm \sqrt{4z^2 - 16(6+z^2)}}{8}$$
$$= -\frac{z}{4} \pm \frac{i}{4}\sqrt{3z^2+24}$$

Thus branch points occur over $z = \pm i\sqrt{8} = \pm 2i\sqrt{2}$, and so at $\left(2i\sqrt{2}, \frac{-i\sqrt{2}}{2}\right)$ and $\left(-2i\sqrt{2}, \frac{+i\sqrt{2}}{2}\right)$. (∞, ∞) is a pinch point, by the theorem.

(Alternatively, put $\tilde{z} = \dfrac{1}{z}$, $\tilde{w} = \dfrac{1}{w}$ and obtain $\tilde{w}^2 + 2\tilde{z}\tilde{w} + 4\tilde{z}^2 + 6\tilde{w}^2\tilde{z}^2 = 0$.
When $\tilde{z} = 0$, $\tilde{w} = 0$ is a double root and gives the pinch point.)

The conic is a double cover of the z-sphere branched at $(2i\sqrt{2}, -i\sqrt{2}/2)$ and $(-2i\sqrt{2}, +i\sqrt{2}/2)$ with (∞, ∞) as a pinch point.

(iii) $A=1$, $B=2$, $C=1$ so $B^2 - 4AC = 0$.
By the theorem the conic is a topological sphere.

$$\text{Now } w = \frac{2z \pm \sqrt{4z^2 - 4(z^2+6z)}}{2}$$
$$= -z \pm i\sqrt{6z}.$$

The only finite branch point is $(z, w) = (0, 0)$.
Since a conic has 0 or 2 branch points, $(z, w) = (\infty, \infty)$ is the other.
The conic is a double cover of the z-sphere branched at (0,0) and (∞, ∞).

Question 14

(i) $\chi = 2 - 2g - b$.

(ii) $\tilde{S} \xrightarrow{n} S$ an unbranched cover.

This implies $\beta(S) \leq \beta(\tilde{S}) \leqslant n\beta(S)$, and $\chi(\tilde{S}) = n\chi(S)$.
Now $\beta(S) = 2$ and $\beta(\tilde{S}) = \tilde{b}$.

$$\chi(\tilde{S}) = 2 - 2(g+k) - \tilde{b} = n\chi(S) = n(2-2g-2)$$
$$= -2ng.$$

Hence $\tilde{b} = 2 + 2ng - 2(g+k)$.

So

$$2 \leqslant 2 + 2ng - 2(g+k) \leqslant 2n$$

thus

$$-2ng + 2g \leqslant -2k \leqslant 2n - 2 - 2ng + 2g;$$

so

$$1 + ng - g - n \leqslant k \leqslant ng - g$$

i.e.

$$(g-1)(n-1) \leqslant k \leqslant g(n-1).$$

(iii) If $n = 3$ and $g > 2$ then

$$2(g-1) \leqslant k \leqslant 2g.$$

Hence $\tilde{g} = g + k \geqslant 2(g-1) + g = 3g - 2$.
If $g > 2$ then $3g - 2 > 2g$.

Question 15

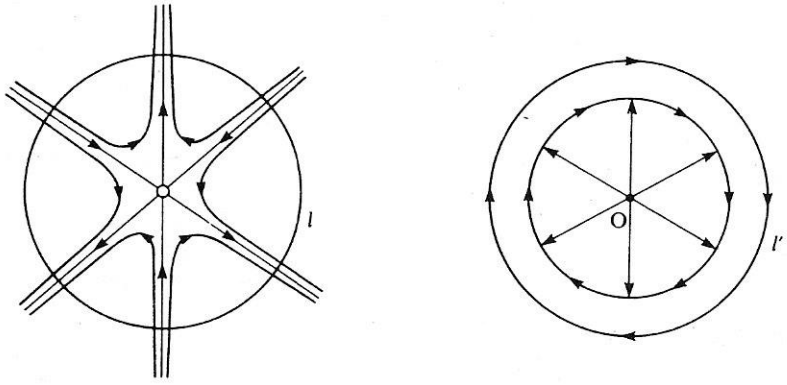
(i) On $C_1 = \{z \in \mathbf{C} : |z| = 1\}$, set $f(z) = z^4 + 2z$, $g(z) = 5$.
Then $|f(z)| = |z^4 + 2z| \leqslant |z|^4 + 2|z| = 3 < 5 = |g(z)|$.
Hence $V_{C_1}(f+g) = V_{C_1}(g) = 0$ and there are no zeros of F inside C_1 (Rouché's Theorem).
On $C_2 = \{z \in \mathbf{C} : |z| = 2\}$, set $f(z) = 2z + 5$, $g(z) = z^4$.
Then $|f(z)| = |2z + 5| \leqslant 2|z| + 5 = 9 < 16 = |g(z)|$.
Hence $V_{C_2}(f+g) = V_{C_2}(g) = 8\pi$ and so F has 4 zeros within C_2 (by Rouché's Theorem).
This proves part (i).

(ii) Consider $F(x)$, $x \in \mathbf{R}$: $F(x) = x^4 + 2x + 5$.
$F'(x) = 4x^3 + 2 = 0$ when $x = \sqrt[3]{-\frac{1}{2}}$, giving an obvious minimum.
Then $F(\sqrt[3]{-\frac{1}{2}})$ is roughly $(-0.8)^4 + 2(-0.8) + 5 > 0$.
so F is never zero on the real axis.
Next consider $z = iy$, in which case $F(z) = y^4 + 2iy + 5$.
This cannot be zero, since when $iy = 0$, $y^4 + 5 = 5$.
Now consider $C_3 = \{z = iy : -2 \leqslant y \leqslant 2\}$
and $C_4 = \{z \in C_2 : \mathrm{Re}(z) \geqslant 0\}$.

$V_{C_3}(F) = V_{C_3}(y^4 + 2iy + 5) < \pi$
$V_{C_4}(F) = 4\pi + \varepsilon$, $|\varepsilon| < \pi$
Thus $V_{C_3 \cup C_4}(F) = V_{C_3}(F) + V_{C_4}(F)$ lies between
$3\pi - |\varepsilon|$ and $5\pi + |\varepsilon|$
(or strictly between 2π and 6π).
Hence $V_{C_3 \cup C_4}(F) = 4\pi$ and F has two zeros in the region $\mathrm{Re}(z) > 0$.
These occur as a complex conjugate pair (since the coefficients of F are real) and hence lie one in the first, and one in the fourth, quadrant. Therefore the remaining two roots lie one in the second and one in the third.

Question 16

(i)



As p describes l once, the 'sharp' end of $V_0(l)$ describes a curve l' of winding number -2 about a fixed point O.

(ii) By the Poincaré theorem, any flow field with a finite number of singularities on a closed orientable surface S has index sum $\chi(S)$. But an orientable surface is determined by χ. Hence S is unique. (In fact, it's a double torus!)